

# VU Research Portal

## Contact and non-contact type Hamiltonian systems generated by second-order Lagrangians

Angenent, S.B.; van den Berg, G.J.B.; van der Vorst, R.C.A.M.

### ***published in***

Ergodic theory and dynamical systems  
2007

### ***DOI (link to publisher)***

[10.1017/S0143385706000605](https://doi.org/10.1017/S0143385706000605)

### ***document version***

Publisher's PDF, also known as Version of record

[Link to publication in VU Research Portal](#)

### ***citation for published version (APA)***

Angenent, S. B., van den Berg, G. J. B., & van der Vorst, R. C. A. M. (2007). Contact and non-contact type Hamiltonian systems generated by second-order Lagrangians. *Ergodic theory and dynamical systems*, 27(1), 23-49. <https://doi.org/10.1017/S0143385706000605>

### **General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal ?

### **Take down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

### **E-mail address:**

[vuresearchportal.ub@vu.nl](mailto:vuresearchportal.ub@vu.nl)

## Contact and non-contact type Hamiltonian systems generated by second-order Lagrangians

S. B. ANGENENT<sup>†</sup>, J. B. VAN DEN BERG<sup>‡</sup> and R. C. A. M. VANDERVORST<sup>‡</sup>

<sup>†</sup> Department of Mathematics, University of Wisconsin-Madison, Madison,  
WI 53706-1388, USA

(e-mail: [angenent@math.wisc.edu](mailto:angenent@math.wisc.edu))

<sup>‡</sup> Mathematics Department, Vrije Universiteit Amsterdam, Amsterdam, The Netherlands

(Received 12 December 2006 and accepted in revised form 13 July 2006)

**Abstract.** We show that some very naturally occurring energy manifolds that are induced by second-order Lagrangians  $L = L(u, u', u'')$  are *not*, in general, of contact type in  $(\mathbb{R}^4, \omega)$ . We also comment on the more general question whether there exist any contact forms on these energy manifolds for which the associated Reeb vector field coincides with the Hamiltonian vector field.

### 1. Introduction

This paper was motivated by the question ‘are the level sets of the Hamiltonian  $H : \mathbb{R}^4 \rightarrow \mathbb{R}$  associated to the variational problem

$$\delta \int \left\{ \frac{1}{2}(u'')^2 + \frac{\alpha}{2}(u')^2 + F(u) \right\} dx = 0$$

of contact type?’.

If the highest-order term  $(u'')^2$  is removed, then the resulting variational problem is the standard ‘ $T - V$ ’ Lagrangian from classical mechanics, whose associated Hamiltonians are well known to have level sets of contact type.

When the  $(u'')^2$  term is present it turns out that the level sets  $M$  of the associated Hamiltonian  $H : \mathbb{R}^4 \rightarrow \mathbb{R}$  are only of contact type for certain values of the parameter  $\alpha$ . We construct here very simple explicit examples of functions  $F : \mathbb{R} \rightarrow \mathbb{R}$  for which  $H^{-1}(0)$  is *not* of contact type for a large range of parameter values  $\alpha$  (see §8.4).

Before stating our results we recall some definitions.

**1.1. Hypersurfaces of contact type.** The energy manifold  $M = H^{-1}(0)$  of a smooth Hamiltonian function  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ , which throughout this paper will assume to be regular (i.e.  $dH \neq 0$  on  $M$ ), is said to be of *contact type* in  $(\mathbb{R}^{2n}, \omega)$  if there exists a one-form  $\theta$  on  $M$  such that

$$d\theta = -j^*(\omega) \quad \text{and} \quad i_{X_H}\theta \neq 0$$

hold on  $M$ . Here  $\omega = \sum dq_i \wedge dp_i$  is the usual symplectic form on  $\mathbb{R}^{2n}$ ,  $j : M \rightarrow \mathbb{R}^{2n}$  is the inclusion map, and  $X_H = (H_p, -H_q)$  is the Hamiltonian vector field on  $M$ . (Note that the same definition can be given for arbitrary symplectic manifolds  $(M^{2n}, \omega)$ .)

Such a contact form  $\theta$  can always be written as  $\theta = \theta_0 + \beta$ , where  $\theta_0 = \mathbf{p} d\mathbf{q} = \sum p_i dq_i$  is the *canonical form* (or *standard Liouville form*), and  $\beta$  is any closed one-form on  $M$ .

The contact type condition first appeared in Weinstein's paper [20] in which he stated his celebrated conjecture to the effect that '*every compact energy manifold  $M$  of contact type carries a closed orbit of the Hamiltonian vector field  $X_H$* '. This was later proved by Viterbo [19]. If  $M$  is of contact type, then the powerful techniques involving pseudoholomorphic curves can be used to study the periodic orbits of  $X_H$  on  $M$ , and in more general settings (see [12]). Without the contact type condition,  $C^\infty$ -smooth counterexamples of energy manifolds in  $(\mathbb{R}^{2n}, \omega)$  which contain no periodic orbits have been given by Ginzburg [6, 7] and Hermann [10] for dimensions  $2n \geq 6$ . Ginzburg and Gürel [8] extended the result to  $\mathbb{R}^4$ , exhibiting a  $C^2$ -smooth example.

Although many Hamiltonian systems from differential geometry and classical mechanics are known to have all of their energy manifolds of contact type, there does not seem to be a clear procedure for deciding whether any given energy manifold  $M \subset \mathbb{R}^{2n}$  is or is not of contact type. In this paper we address this issue for Hamiltonian systems on  $\mathbb{R}^4$  which arise from second-order variational problems of the type

$$\delta \int L(u, u', u'') dt = 0. \quad (1)$$

Given any vector field  $X$  on an odd-dimensional manifold  $M$  (such as  $X_H$  on  $M$ ), one may also ask the more general question 'is there a contact form  $\lambda$  on  $M$  such that the vector field  $X$  (after renormalization) is the Reeb vector field of  $\lambda$ ?'. If  $M$  is of contact type, then the answer is 'yes', and one can simply put  $\lambda = \theta$ . The question 'is  $M$  of contact type' is more restrictive since the symplectic form  $d\lambda$  corresponding to the contact form one looks for is prescribed (it should be  $j^*\omega$ ).

**1.2. Second-order Lagrangians.** A second-order Lagrangian is a function of the form  $L = L(u, u', u'')$ , where  $u = u(t)$  is a scalar function on  $\mathbb{R}$ . Such a Lagrangian is assumed to be convex in the  $u''$ -variable. The Euler–Poisson† equation for the variational problem (1) is a fourth-order ordinary differential equation (ODE), and is given by

$$\frac{d^2}{dt^2} \frac{\partial L}{\partial u''} - \frac{d}{dt} \frac{\partial L}{\partial u'} + \frac{\partial L}{\partial u} = 0. \quad (2)$$

This ODE is equivalent to a Hamiltonian system on  $(\mathbb{R}^4, \omega)$  with Hamiltonian function given by

$$H = p_u v + L^*(u, v, p_v). \quad (3)$$

† These are the equations obtained by requiring the first variation of the action to vanish along all possible variations  $u \mapsto u + \epsilon \varphi$ . In [1], these equations are called the Euler–Poisson equations rather than the Euler–Lagrange equations.

Here  $L^*(u, v, p_v) = \sup_{w \in \mathbb{R}} p_v w - L(u, v, w)$  is the Legendre transform of  $L(u, v, w)$  with respect to the third variable. The correspondence between the canonical coordinates  $x = (u, v, p_u, p_v)$  and  $u$  and its derivatives is as follows:

$$v = u', \quad p_u = -\frac{d}{dt} \frac{\partial L}{\partial u''} + \frac{\partial L}{\partial u'} \quad \text{and} \quad p_v = \frac{\partial L}{\partial u''}.$$

In §3.1 this procedure is explained in more detail for general higher-order Lagrangians. See also [1].

Due to the  $p_u v$  term in the definition of the Hamiltonian (3), the energy manifolds  $M = H^{-1}(0)$  are always *non-compact*<sup>†</sup>. The particular question we shall be concerned with is whether or not the zero energy manifold  $M$  is of contact type in general. The following result identifies a large class of second-order Lagrangians for this is indeed the case.

**THEOREM 1.1.** *If  $L(u, v, w) > 0$  for all  $(u, v, w) \in \mathbb{R}^3$ , then  $M$  is of contact type and the canonical form  $\theta_0 = p_u du + p_v dv$  is a contact form on  $M$ .*

This holds in much greater generality for Hamiltonian systems coming from variational problems of the type  $\delta \int L(u, u', \dots, u^{(n)}) dt = 0$ , where the ‘Lagrangian’  $L$  is strictly positive. See §3.1 for the proof.

We continue now with a special class of Lagrangians which are not necessarily positive, and we show that among these there are Lagrangians for which  $M$  is not of contact type.

**1.3. The Swift–Hohenberg and extended Fisher Kolmogorov (eFK) models.** As a special case, one considers Lagrangians of the form

$$L(u, v, w) = \frac{1}{2}w^2 + \frac{\alpha}{2}v^2 + F(u). \quad (4)$$

The associated Hamiltonian is then given by  $H(x) = p_u v + \frac{1}{2}p_v^2 - (\alpha/2)v^2 - F(u)$ , and the Euler–Poisson equation for the variational problem is fourth order:

$$\frac{d^4 u}{dt^4} - \alpha \frac{d^2 u}{dt^2} + F'(u) = 0.$$

For various different choices of the ‘potential’  $F(u)$  and parameter  $\alpha$ , this equation is known in the mathematical physics literature as the extended Fisher Kolmogorov (eFK) or Swift–Hohenberg equation. See [9, 17], [4, Introduction] and the references given there.

Clearly, if the potential  $F$  is positive and the parameter  $\alpha \geq 0$ , then Theorem 1.1 implies that  $M$  is of contact type. However, for potentials that are not strictly positive, or for negative values of the parameter  $\alpha$ , the question becomes more delicate, and the geometry and topology of  $M$  will come into play. The next three theorems summarize cases where  $M$  is again of contact type, but  $L$  is not necessarily strictly positive.

**THEOREM 1.2.** *If  $\alpha \geq 0$  and  $F(u)$  only has simple zeroes, then  $M$  is of contact type, with  $\theta = \theta_0 + d(vp_v) + \beta$  as contact form, and where  $\beta$  is a closed form which can be chosen arbitrarily small in  $C^\infty$ .*

<sup>†</sup> We may restrict to just the zero energy manifolds since all other energy manifolds may be obtained by simply replacing  $L$  by  $L + E$ .

The proof is given in §4.

In §5 we recall that for certain choices of  $F$  the flow on  $M$  admits a global Poincaré section. This additional structure also allows one to construct contact forms compatible with  $\omega$ , and is useful for the study of periodic orbits of  $X_H$  on  $M$ .

**THEOREM 1.3.** *If  $F(u) > 0$  and  $F(u)/u^2 \rightarrow \infty$  as  $u \rightarrow \pm\infty$ , then there exists a diffeomorphism  $\Lambda : M \rightarrow \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$  which carries trajectories of  $X_H$  to solutions of a non-autonomous planar Hamiltonian system.*

*If, in addition,  $\alpha < 0$ , then the growth condition on  $F(u)$  is not needed, and one can exhibit an explicit diffeomorphism  $\Lambda$ .*

Poincaré sections can be found in much more general settings for second-order Lagrangians. In this theorem we chose to restrict the details to the Swift–Hohenberg and eFK models for simplicity. See §5 and [3] for a more detailed account on this subject.

Here a non-autonomous planar Hamiltonian system refers to the case in which  $M = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$  with coordinates  $(p, q) \in \mathbb{R}^2$ ,  $t \in (\mathbb{R}/\mathbb{Z})$  and  $\lambda = p dq - H(p, q, t) dt$ . Such a system is not necessarily of contact type as we show by an example in §5.3. The following is therefore not an immediate corollary of Theorem 1.3.

**THEOREM 1.4.** *If  $F(u) > 0$  and if  $F'(u)/u \rightarrow \infty$  as  $u \rightarrow \pm\infty$ , then  $M$  is of contact type.*

**1.4. Non-contact type energy manifolds.** So far we have only given sufficient conditions for  $M$  to be of contact type. These conditions do not cover all cases, in particular the situation in which  $F(u)$  changes sign and  $\alpha < 0$  is not covered. In §8.4 we show that the energy manifold  $M$  is not, in fact, always of contact type in  $(\mathbb{R}^4, \omega)$ .

**THEOREM 1.5.** *There exists a potential  $F : \mathbb{R} \rightarrow \mathbb{R}$  and an  $\alpha_* < 0$  such that for all  $\alpha < \alpha_*$  the energy manifold  $M$  is not of contact type.*

In §8.4 we describe for which shapes of  $F$  the above theorem holds. This leads to a very large class of non-contact type energy manifolds. The problem of finding energy manifold which are not of contact type has been approached from a different perspective in [5].

All of this leaves us with a few unanswered questions of which we mention two.

**QUESTION 1.** *For the Swift–Hohenberg equation with  $F(u) = \frac{1}{4}(1 - u^2)^2 + E$  (a usual choice in physics models) for  $E < 0$  and  $\alpha < 0$  we still do not know whether  $M$  is of contact type or not, i.e. is the hypersurface in  $\mathbb{R}^4$  defined by*

$$p_u v + \frac{1}{2} p_v^2 - \frac{\alpha}{2} v^2 - \frac{1}{4} (1 - u^2)^2 - E = 0$$

*of contact type?*

**QUESTION 2.** *If one chooses  $F(u)$  as in Theorem 1.5, then for sufficiently large negative  $\alpha$  the hypersurface  $M$  is not of contact type, while for  $\alpha > 0$  it is of contact type. How does the transition from contact to non-contact type take place as  $\alpha$  decreases from  $+\infty$  to  $-\infty$ ?*

## 2. Representation of $X_H$ by an arbitrary Reeb vector field

Any one-form  $\theta$  on a  $(2n - 1)$ -dimensional manifold defines a variational problem for closed immersed curves  $\gamma \subset M$  in which one requires the *action*

$$A(\gamma) \stackrel{\text{def}}{=} \int_{\gamma} \theta,$$

to be stationary. If  $X$  is any vector field along  $\gamma$ , then the variation of the action by  $X$  is

$$dA(\gamma) \cdot X = \int_{\gamma} i_X d\theta = \int_0^1 d\theta(X(t), \gamma'(t)) dt.$$

Hence, the action will be stationary at  $\gamma$  if and only if  $i_{\gamma'(t)} d\theta = 0$  holds. See [2].

We will call the one-form  $\theta$  *non-degenerate* if  $d\theta$  has maximal rank everywhere. If  $\theta$  is non-degenerate then

$$\mathcal{L}_p = \ker d\theta \stackrel{\text{def}}{=} \{Y \in T_p M \mid i_Y d\theta = 0\}, \quad (5)$$

is one dimensional for all  $p \in M$  and thus defines a linebundle (direction field) on  $M$ . In this case, the stationary curves for  $A(\gamma)$  are precisely the integral curves for the line bundle  $\mathcal{L}$ . The non-degeneracy condition for  $\theta$  is the analog for the variational problem  $\delta \int_{\gamma} \theta = 0$  of the Legendre–Hadamard condition from the calculus of variations.

If the manifold  $M$  is oriented then  $\mathcal{L}$  has a nowhere vanishing section<sup>†</sup>. We will call any positively oriented section of the line bundle  $\mathcal{L}$  a *pseudo-Reeb vector field* for the one-form  $\theta$ . Critical points of the action  $A(\gamma)$  are then closed orbits of any pseudo-Reeb vector field for  $\theta$ .

If the form  $\theta$  is a contact form, i.e. if  $\theta \wedge (d\theta)^{n-1} \neq 0$  everywhere, then  $d\theta$  clearly has maximal rank everywhere, so contact forms are non-degenerate. For a contact form  $\theta$  there is a chosen section  $X$  of  $\mathcal{L}$ , defined by  $i_X \theta \equiv 1$ . This pseudo-Reeb vector field is called the *Reeb vector field* of  $\theta$ . Conversely, if  $\theta$  is non-degenerate and if there is a pseudo-Reeb vector field  $X$  such that  $i_X \theta > 0$ , then  $\theta$  is a contact form.

For Hamiltonian systems in  $\mathbb{R}^{2n}$ , i.e. if  $M = H^{-1}(0)$  with 0 a regular value of  $H$ , the form  $\theta_0 = \mathbf{p} \cdot d\mathbf{q}$  is non-degenerate, and the Hamiltonian vector field  $X_H$  is a pseudo-Reeb vector field for  $\theta_0$  on  $M$ .

The energy surface  $M$  will be of contact type if one can find a closed form  $\beta$  on  $M$  such that  $\theta_0 + \beta$  is a contact form. The Reeb vector field for  $\theta_0 + \beta$  is then a multiple of the Hamiltonian vector field  $X_H$ . The more general question that can be asked is: ‘Does there exist any contact form  $\theta$ , such that the Hamiltonian vector field  $X_H$  is a pseudo-Reeb vector field for  $\theta$ ?’ This question leaves more freedom in choosing  $\theta$  since the condition  $d\theta = -j^*(\omega)$  is omitted.

Although we do not give any positive or negative results on the more general question, the following observations seem to indicate that the situation in which a Hamiltonian vector field  $X_H$  is a Reeb vector field for a contact form  $\lambda$  with  $d\lambda \neq c j^* \omega$ , for any constant  $c \neq 0$ , is unusual.

<sup>†</sup> Indeed, at each point  $p \in M$  the quotient  $T_p M / \mathcal{L}_p$  is oriented by the volume form  $(d\theta)^{n-1}$ , while the tangent space  $T_p M$  is assumed to have an orientation. These two orientations induce an orientation on  $\mathcal{L}_p$ .

LEMMA 2.1. *If  $X_H$  is a pseudo-Reeb vector field for a contact form  $\lambda$  on  $M$ , then  $d\lambda = fj^*\omega$  for some smooth nowhere vanishing function  $f : M \rightarrow \mathbb{R}$ . Furthermore, this function  $f$  is a conserved quantity for  $X_H$ :  $X_H(f) = 0$ .*

*There is a unique vector field  $Y$  on  $M$  which satisfies  $i_Y(d\lambda) = df$  and  $i_Y\lambda = 0$ .*

*Let  $X$  be the Reeb vector field of  $\lambda$  (i.e.  $X$  is a multiple of  $X_H$  which satisfies  $\lambda(X) = 1$ ). Then the vector fields  $X$  and  $Y$  commute:  $[X, Y] = 0$ .*

*Wherever  $Y \neq 0$  the vector fields  $X$  and  $Y$  are linearly independent.*

*Proof.* Since both  $i_{X_H}j^*\omega = 0$  and  $i_{X_H}d\lambda = 0$  we have  $\ker j^*\omega = \ker d\lambda$ . Let  $\ker j^*\omega = \text{span}(X)$ , and write  $\xi = \xi_1 X + \xi_2 Y + \xi_3 Z$  and  $\eta = \eta_1 X + \eta_2 Y + \eta_3 Z$ , where  $\{X, Y, Z\}$  is a basis for  $TM$ , normalized by  $\omega(Y, Z) = 1$ . Write  $\tilde{\omega} = d\lambda$ . Then  $\tilde{\omega}(\xi, \eta) = (\xi_2\eta_3 - \xi_3\eta_2)\tilde{\omega}(Y, Z)$ , and  $\omega(\xi, \eta) = (\xi_2\eta_3 - \xi_3\eta_2)\omega(Y, Z) = \xi_2\eta_3 - \xi_3\eta_2$ . Thus, we find  $d\lambda = fj^*\omega$ , with  $f = \tilde{\omega}(Y, Z)$ .

Since  $fj^*\omega = d\lambda$  is closed it follows that  $df \wedge j^*\omega = 0$  and, thus,

$$0 = i_{X_H}(df \wedge j^*\omega) = (i_{X_H}df)j^*\omega - df \wedge i_{X_H}j^*\omega = (i_{X_H}df)j^*\omega,$$

which implies that  $i_{X_H}df = 0$ . The function  $f$  is therefore an integral of the vector field  $X_H$ .

Since  $\ker df \supset \ker d\lambda$ , a vector  $Y$  exists such that  $df(Z) = d\lambda(Y, Z)$  for all  $Z \in T_pM$ . Two different choices of  $Y$  differ by an element of  $\ker d\lambda$  while at most one  $Y$  can satisfy  $\lambda(Y) = 0$ .

If  $X$  is the Reeb vector field for  $\lambda$ , then  $\mathcal{L}_X\lambda = 0$ . Hence,  $\lambda(\mathcal{L}_X(Y)) = \mathcal{L}_X(\lambda(Y)) = 0$  so that  $[X, Y] = \mathcal{L}_XY$  belongs to  $\ker \lambda$ .

One also has, using  $\mathcal{L}_X\lambda = 0$  and  $i_Y d\lambda = df$ ,

$$i_{\mathcal{L}_XY}d\lambda = \mathcal{L}_X(i_Y d\lambda) - i_Y d\mathcal{L}_X\lambda = \mathcal{L}_X(df) = d\mathcal{L}_Xf = 0.$$

Since  $\mathcal{L}_XY$  belongs both to  $\ker \lambda$  and to  $\ker d\lambda$ , we have  $\mathcal{L}_XY = 0$  and, hence,  $X$  and  $Y$  commute.

If  $Y \neq 0$  at some  $p \in M$ , then at that point one has  $i_Y d\lambda = df \neq 0$  and  $i_X d\lambda = 0$ , so that  $X$  and  $Y$  cannot be linearly dependent at  $p$ .  $\square$

It is well known that a second integral of the motion can severely restrict the possible dynamics of  $X_H$  on  $M$ . For instance, any periodic orbit of  $X$  on which  $Y \neq 0$  must appear in a family of periodic orbits of  $X$ . Also, for  $n = 2$  if  $f : M \rightarrow \mathbb{R}$  has a compact regular level surface  $S = f^{-1}(c)$ , then this must be a 2-torus, and the flow of  $X_H$  will be the standard linear flow. By the implicit function theorem the same will apply to  $S' = f^{-1}(c')$  for  $c'$  close to  $c$ , so that an open subset of  $M$  is foliated by invariant tori with linear flow.

A typical Poincaré plot (see Figure 1) for the Swift–Hohenberg equation shows none of these phenomena, suggesting that, generally,  $X_H$  will only be a pseudo-Reeb vector field for one particular form  $\lambda$ .

### 3. Generalities about finding contact forms

3.1. *Positive Lagrangians.* As explained in [1], the Euler–Lagrange–Poisson equations of a general  $n$ th-order Lagrangian variational problem of the type

$$\delta \int L(u, u', u'', \dots, u^{(n)}) dt = 0 \quad (6)$$

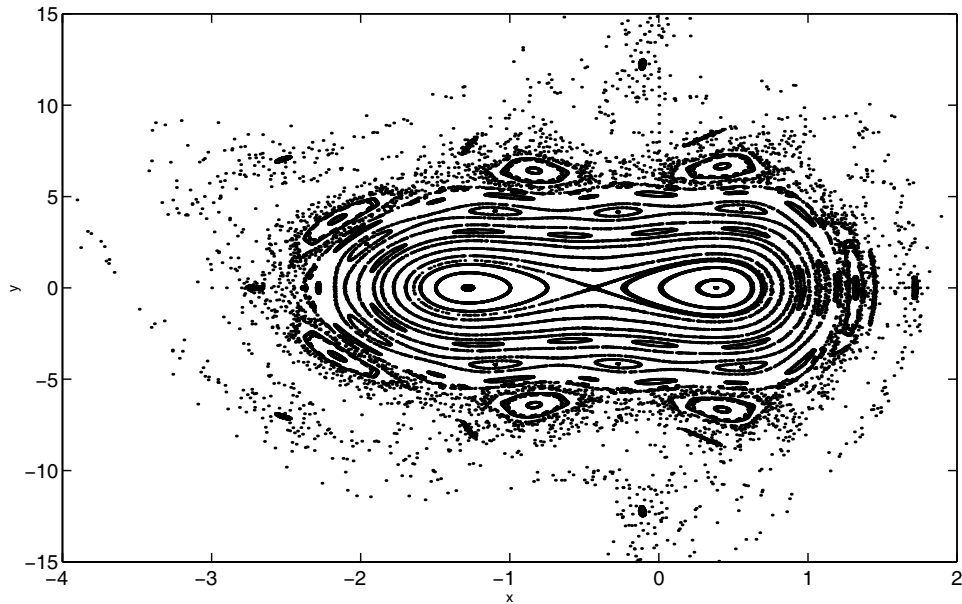


FIGURE 1. A Poincaré plot for parameter values  $\alpha = -10$  and  $E = 10$ . Reproduced with permission from [4].

whose ‘Lagrangian’  $L = L(u_0, u_1, \dots, u_n)$  is strictly convex in the highest derivative  $u_n$ , can be transformed to a Hamiltonian system on the cotangent bundle of the space  $J^{n-1}$  of  $(n-1)$ -jets of functions  $u : \mathbb{R} \rightarrow \mathbb{R}$ . Alternatively, one writes  $u_0, \dots, u_n$  for the first  $n$  derivatives of  $u$  and introduces variables  $p_0, \dots, p_{n-1}$ , where the last is related to  $u_0, \dots, u_n$  by

$$p_{n-1} = \frac{\partial L(u_0, \dots, u_n)}{\partial u_n}, \quad (7)$$

due to the assumption that  $\partial_{u_n}^2 L > 0$ . One then defines the Legendre transform by

$$L^*(u_0, u_1, \dots, u_{n-1}, p_{n-1}) = \sup_{w \in \mathbb{R}} \{p_{n-1}w - L(u_0, u_1, \dots, u_{n-1}, w)\}$$

and the ‘Hamiltonian’ by

$$H(u_0, \dots, u_{n-1}, p_0, \dots, p_{n-1}) = p_0 u_1 + \dots + p_{n-2} u_{n-1} + L^*(u_0, u_1, \dots, u_{n-1}, p_{n-1}).$$

The Hamilton equations for  $H$  give the extremals for (6).

**LEMMA 3.1.** *If one defines  $\mathbf{M}$  to be the zero energy manifold  $H^{-1}(0) \subset \mathbb{R}^{2n}$  of  $H$ , and if the Lagrangian  $L$  is strictly positive, then  $\mathbf{M}$  is of contact type with the standard contact form  $\theta_0 = p_0 du_0 + \dots + p_{n-1} du_{n-1}$ .*

*Proof.* Let  $p_{n-1}$  and  $u_n$  be related by (7), then

$$u_n = \frac{\partial L^*(u_0, \dots, u_{n-1}, p_{n-1})}{\partial p_{n-1}}$$

and  $L + L^* = p_{n-1} u_n$ . One has  $i_{X_H} \theta_0 = p_0 u'_0 + \dots + p_{n-1} u'_{n-1}$ , and from the Euler–Lagrange–Poisson equations one finds that  $u'_i = u'_{i+1}$  for  $i < n-1$  and  $u'_{n-1} = \partial L^* / \partial p_{n-1} = u_n$ . Therefore,



$$\begin{aligned}
i_{X_H}\theta_0 &= p_0u_1 + \cdots + p_{n-2}u_{n-1} + p_{n-1}\frac{\partial L^*}{\partial p_{n-1}} \\
&= p_0u_1 + \cdots + p_{n-2}u_{n-1} + L + L^* \\
&= H + L.
\end{aligned}$$

Since  $H = 0$  on  $M$  we find  $i_{X_H}(\theta_0) = L > 0$  on  $M$ , so that  $\theta_0$  is a contact form on  $M$ , whose Reeb vector field coincides with  $X_H$ .  $\square$

This lemma, when applied to second-order Lagrangians, proves Theorem 1.1.

**3.2. A necessary condition for contact type.** In [20], Weinstein gave an example of a compact hypersurface which is not of contact type, together with a general necessary condition for a compact hypersurface with  $H^1(M; \mathbb{R}) = 0$  to be of contact type. See also [11]. From these sources we may distill the following variation on Weinstein's criterion. Let  $M = H^{-1}(0)$  be an energy manifold of a Hamiltonian system on  $(\mathbb{R}^{2n}, \omega)$ , with  $\omega$  the standard symplectic form $\dagger$ .

**LEMMA 3.2.** *Given two contractable periodic orbits  $\gamma_1, \gamma_2 \subset M$  of  $X_H$  for which the standard contact form  $\theta_0$  yields actions of opposite signs, then one may conclude that  $M$  is not of contact type.*

*Proof.* Any possible contact form  $\theta$  is of the form  $\theta = \theta_0 + \beta$ , with  $d\beta = 0$ .

Represent the curves  $\gamma_i$  as boundaries of discs,  $\gamma_i = \partial\Delta_i$ ,  $\Delta_i \subset M$ . Then by Stokes' theorem one has

$$\oint_{\gamma_i} \theta = - \int_{\Delta_i} \omega = \oint_{\gamma_i} \theta_0.$$

If  $\theta$  were a contact form, then both actions  $\oint_{\gamma_i} \theta$  would have the same sign, and hence both integrals  $\oint_{\gamma_i} \theta_0$  would also have the same sign, a contradiction.  $\square$

**3.3. Fixing almost contact forms.** In [11], Hofer and Zehnder show that all energy manifolds of any classical mechanical system on  $\mathbb{R}^{2n}$  with Hamiltonian  $H(q, p) = \frac{1}{2}|p|^2 + V(q)$  are of contact type. They do this by first observing that the canonical form satisfies  $i_X\theta_0 \geq 0$  and then perturbing the form  $\theta_0$  to achieve strict inequality. If one replaces the explicit perturbation in [11] by something more abstract, one arrives at the following result $\ddagger$ .

**LEMMA 3.3.** *Let  $M$  be a  $(2n - 1)$ -dimensional manifold with a non-degenerate one-form  $\lambda$ . Let  $X$  be a pseudo-Reeb vector field for  $\lambda$ . Assume that  $i_X\lambda \geq 0$ , and also that the set  $S = \{p \in M \mid i_X\lambda(p) = 0\}$  satisfies*

$$\text{for all } p \in S, \text{ there exists } t_- < 0 < t_+ : \Phi_{t_{\pm}}(p) \notin S.$$

$\dagger$  As the referee has pointed out to us, McDuff [15, Theorem 5.2], using the ideas of Sullivan [18], has proved a necessary and sufficient condition for compact odd-dimensional manifold with a non-degenerate exact two-form  $d\alpha$  to be of contact type. In the compact case, it is a refinement of our Lemma 3.2. For the concrete non-compact examples we are dealing, with it does not seem obvious to us how one would use the extra information contained in McDuff's theorem.

$\ddagger$  As observed by the referee, this lemma is closely related to a theorem of Sullivan [18], a self-contained account of which is given in [14].

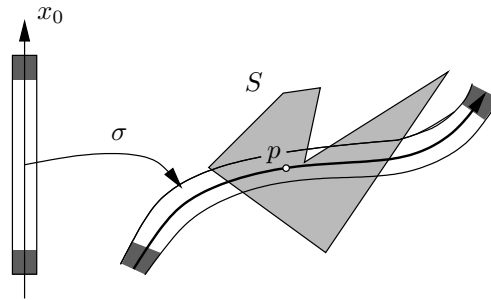


FIGURE 2. Proof of Lemma 3.3.

Then there exists a smooth function  $f : M \rightarrow \mathbb{R}$  such that  $\lambda^* = \lambda + \epsilon df$  is a contact form on  $M$  for all  $\epsilon \in (0, 1)$ , and for which  $X$  is a pseudo-Reeb vector field ( $i_X d\lambda^* = 0$ ).

*Proof.* For any given  $p \in S$  we choose a parameterized  $2n-2$  ball  $\sigma_0 : B_r^{2n-2} \rightarrow M$  which is transverse to the vector field  $X$ . Then  $\sigma(x_0, x_1, \dots, x_{2n-2}) = \Phi^{x_0}(\sigma_0(x_1, \dots, x_{2n-2}))$  is a local diffeomorphism which straightens the flow, i.e. it maps lines parallel to the  $x_0$  axis to flow lines of the vector field  $X$ , and it maps  $\partial/\partial x_0$  to  $X$ . It is a diffeomorphism on  $B_r^{2n-2} \times [t_-, t_+]$  if  $r > 0$  is small enough.

Since  $\Phi^{\pm}(p)$  does not belong to the closed set  $S$ , one can choose  $r > 0$  so small that  $\sigma(x_0, \dots, x_{2n-2})$  does not lie in  $S$  if  $x_1^2 + \dots + x_{2n-2}^2 \leq r^2$  and  $x_0 \in [t_-, t_- + r] \cup [t_+ - r, t_+]$ . We also choose  $r$  so small that  $t_- + r < 0 < t_+ - r$ . Now let  $0 \leq \eta \in C^\infty(\mathbb{R})$  satisfy  $\eta(x_0) = 0$  for  $x_0 \leq t_-$  and  $x_0 \geq t_+$ , as well as  $\eta(x_0) = 1$  for  $x_0 \in [t_- + r, t_+ - r]$ . In addition we pick a  $0 \leq \zeta \in C^\infty(\mathbb{R}^{2n-2})$ , which is supported in  $B_r^{2n-2}$ , and with  $\zeta(0) > 0$ . We then define  $f : M \rightarrow \mathbb{R}$  by  $f \circ \sigma(x_0, \dots, x_{2n-2}) = x_0 \eta(x_0) \zeta(x_1, \dots, x_{2n-2})$  on  $\sigma(B_r^{2n-2} \times [t_-, t_+])$  and  $f = 0$  elsewhere.

Using

$$i_X df = \frac{\partial x_0 \eta(x_0) \zeta(x_1, \dots, x_{2n-2})}{\partial x_0}$$

one now easily verifies that  $i_X df \geq 0$  on  $S$ , and  $i_X df > 0$  at  $p$  and by continuity in a neighborhood  $N_p$  of  $p$ . One also sees that  $i_X df < 0$  only in  $\sigma([t_-, t_- + r] \times B_r^{2n-2}) \cup \sigma([t_+ - r, t_+] \times B_r^{2n-2})$ , i.e.  $i_X df < 0$  outside of some neighborhood of  $S$ .

Let  $\{p_k\}_{k \in \mathbb{N}}$  be a sequence of points for which the neighborhoods  $N_{p_i}$  cover  $S$ . Denote the functions obtained by the above construction by  $f_k$ . Then for each  $k \in \mathbb{N}$  the quantity

$$A_k = \sup_{M \setminus S} \frac{(-i_X df_k)_+}{i_X \lambda}$$

is finite, since  $i_X df_k > 0$  on a neighborhood of  $S$ . We define

$$F = \sum_{k \in \mathbb{N}} \frac{2^{-k} f_k}{A_k + \|f_k\|_{C^k}}.$$

This series converges in  $C^\infty$ . Its sum satisfies  $i_X dF > 0$  on  $S$  and

$$\begin{aligned} -i_X dF &\leq \sum_{k \in \mathbb{N}} 2^{-k} \frac{(-i_X df_k)_+}{A_k} \\ &\leq \sum_{k \in \mathbb{N}} 2^{-k} i_X \lambda = i_X \lambda \end{aligned}$$

on  $M \setminus S$ . Hence,  $i_X(\lambda + \epsilon dF) > 0$  on all of  $M$  for all  $0 < \epsilon < 1$ , as claimed.  $\square$

3.4. *Fixing contact forms without recurrence.* Our proof of Theorem 1.4 is based on the following general observation.

LEMMA 3.4. *Let  $M$  be a  $(2n - 1)$ -dimensional manifold with a non-degenerate one-form  $\lambda$ , and let  $X$  be a pseudo-Reeb vector field for  $\lambda$ . If there is a largest compact invariant set  $K \subset M$  for the flow of  $X$ , and if  $i_X \lambda > 0$  on  $K$ , then a function  $f \in C^\infty(M)$  exists for which  $\lambda^* = \lambda + df$  is a contact form and for which  $X$  is a pseudo-Reeb vector field ( $i_X d\lambda^* = 0$ ).*

*Proof.* Let  $K = K_0 \subset K_1 \subset K_2 \cdots$  be a sequence of compact subsets of  $M$  such that  $K_i$  is in the interior of  $K_{i+1}$  and  $\bigcup_{i \in \mathbb{N}} K_i = M$ . By induction we will construct a sequence of functions  $f_i \in C^\infty(M)$  such that

$$\begin{aligned} \text{the one-forms } \lambda_0 = \lambda, \lambda_i = \lambda_{i-1} + df_i \text{ (} i \geq 1 \text{) are contact forms on } K_i, \\ \text{and } f_i \equiv 0 \text{ on a neighborhood of } K_{i-1}. \end{aligned} \quad (8)$$

The functions  $f_j$  with  $j \geq i + 1$  then all vanish on  $K_i$ , so that  $f = d\sum_{i \in \mathbb{N}} f_i$  is well-defined, and  $\lambda_* = \lambda + df$  is a contact form.

Let  $f_0, \dots, f_i$  be constructed, so that  $\lambda_i$  is a contact form on  $K_i$ , i.e.  $i_X(\lambda_i) > 0$  on  $K_i$ . By continuity there is an open set  $K_i \subset U_i \subset K_{i+1}$  such that  $i_X(\lambda_i) > 0$  on  $U_i$ .

PROPOSITION 3.1. *For each  $p \in K_{i+1} \setminus U_i$  a function  $g_p \in C_c^\infty(M)$  exists such that  $\text{supp } g_p \subset M \setminus K_i$ ,  $i_X dg_p \geq 0$  everywhere, and  $i_X dg_p > 0$  in  $p$ .*

*Proof.* Let  $p \in K_{i+1} \setminus U_i$  be given. The orbit  $\Gamma_p = \{\Phi^t(p) \mid t \in \mathbb{R}\}$  cannot be contained in  $K_{i+1}$ , as if it was, then the closure of  $\Gamma_p$  would be a compact invariant set containing points outside of  $K = K_0$ , thereby contradicting our assumption that  $K$  is the largest compact invariant set. Hence, for some  $t_+ \neq 0$  one has  $\Phi^{t_+}(p) \notin K_{i+1}$ . Without loss of generality, we assume that  $t_+ > 0$ . It is also impossible for the half-orbit  $\Gamma_p^- = \Phi^{(-\infty, 0]}(p)$  to be contained in  $K_{i+1} \setminus U_i$ . If this were to happen, then the  $\alpha$ -limit of  $\Gamma_p^-$  would be a compact invariant subset of the flow outside of  $K_0$ . Thus, a  $t_- < 0$  exists with  $\Phi^{t_-}(p) \in U_i$  or  $\Phi^{t_-}(p) \in M \setminus K_{i+1}$ . In either case we can choose  $t_-$  so that  $\Phi^{[t_-, 0]}(p)$  is disjoint from  $K_i$ .

We straighten the flow in a neighborhood of the orbit segment  $\Phi^{[t_-, t_+]}(p)$ . To this end choose a smooth immersion  $\sigma : \mathbb{R}^{2n-2} \rightarrow M$  with  $\sigma(0) = p$  and which meets the orbit  $\Gamma_p$  transversally at this point. Then let  $\phi : [t_-, t_+] \times \mathbb{R}^{2n-2} \rightarrow M$  be given by

$$\phi(x_0, x_1, \dots, x_{2n-2}) = \Phi^{x_0}[\sigma(x_1, \dots, x_{2n-2})].$$

For sufficiently small  $r > 0$ , this map is a diffeomorphism from  $[t_-, t_+] \times B_r^{2n-2}$  into  $M$ .

Since  $\phi(t_+, 0, \dots, 0) = \Phi^{t_+}(p) \notin K_{i+1}$  we can choose  $\epsilon, r > 0$  so small that  $\phi([t_+ - \epsilon, t_+] \times B_r^{2n-2})$  is disjoint from  $K_{i+1}$ .

Choose  $\eta \in C_c^\infty(t_-, t_+)$  and  $\zeta \in C_c^\infty(B_r^{2n-2})$  and define  $g_p$  by requiring

$$g_p(\phi(x_0, x_1, \dots, x_{2n-2})) = \eta(x_0)\zeta(x_1, \dots, x_{2n-2}),$$

on the tubular neighborhood  $\phi([t_-, t_+] \times B_r^{2n-2})$ , and  $g \equiv 0$  elsewhere. One then has  $i_X dg_p = \eta'(x_0)\zeta(x_1, \dots, x_{2n-2})$ . We now choose  $\zeta \geq 0$  and  $\eta'(x_0) \geq 0$  for  $t_- \leq x_0 \leq t_+ - \epsilon$ , so that  $i_X dg_p \geq 0$  on  $K_{i+1}$ . Finally, we also choose  $\zeta$  and  $\eta$  so that  $\zeta(0, \dots, 0) > 0$  and  $\eta'(0) > 0$ , i.e. so that  $i_X dg_p > 0$  at  $p$ .  $\square$

We complete the proof of Lemma 3.4. For each  $p \in K_{i+1} \setminus U_i$  a  $k_p > 0$  exists such that  $i_X(\lambda_i + k_p dg_p) > 0$  at  $p$ . By continuity this also holds on some small neighborhood  $N_p$  of  $p$ . A finite number of such neighborhoods  $N_{p_1}, \dots, N_{p_m}$  cover  $K_{i+1} \setminus U_i$  and the function  $f_{i+1} = \sum_j k_{p_j} g_{p_j}$  satisfies our requirements in (8).  $\square$

#### 4. Proof of Theorem 1.2

We try one-forms of the form

$$\begin{aligned}\lambda &= \theta_0 + \beta_1 d(up_u) + \beta_2 d(vp_v) \\ &= (\beta_1 + 1)p_u du + \beta_1 u dp_u + (\beta_2 + 1)p_v dv + \beta_2 v dp_v\end{aligned}$$

on  $M = H^{-1}(0)$ , where  $H = p_u v + L^*(u, v, p_v)$ .

We then have

$$\begin{aligned}i_{X_H}\lambda &= (\beta_1 + 1)p_u u' + \beta_1 u p'_u + (\beta_2 + 1)p_v v' + \beta_2 v p'_v \\ &= (\beta_1 + 1)p_u v + \beta_1 u \frac{\partial L}{\partial u} + (\beta_2 + 1)p_v \frac{\partial L^*}{\partial p_v} + \beta_2 v \left(-p_u + \frac{\partial L}{\partial v}\right) \\ &= (\beta_1 - \beta_2 + 1)p_u v + \beta_1 u \frac{\partial L}{\partial u} + \beta_2 v \frac{\partial L}{\partial v} + (\beta_2 + 1)w \frac{\partial L}{\partial w} \\ &= -(\beta_1 - \beta_2 + 1)L^* + \beta_1 u \frac{\partial L}{\partial u} + \beta_2 v \frac{\partial L}{\partial v} + (\beta_2 + 1)w \frac{\partial L}{\partial w} \\ &= (\beta_1 - \beta_2 + 1)L + \beta_1 u \frac{\partial L}{\partial u} + \beta_2 v \frac{\partial L}{\partial v} + (-\beta_1 + 2\beta_2)w \frac{\partial L}{\partial w}.\end{aligned}$$

If  $(\beta_1 - \beta_2 + 1)L + \beta_1 u L_u + \beta_2 v L_v + (-\beta_1 + 2\beta_2)w L_w \geq 0$ , for some constants  $\beta_1$  and  $\beta_2$ , then we see that  $i_X \lambda \geq 0$  everywhere. This generalizes the calculation of §3.1, where  $\beta_1 = \beta_2 = 0$ . Different choices of  $\beta_i$  lead to different ‘starshapedness-like’ conditions on  $L$ . For instance, if one puts  $\beta_1 = \beta_2 = \gamma^{-1} > 0$ , then one finds that

$$\gamma L + u L_u + v L_v + w L_w > 0,$$

for all  $(u, v, w)$ , implies that  $\lambda$  is a contact form and that  $M$  is of contact type.

We will apply this more general criterion now to the Lagrangians defined by (4). Taking  $\beta_1 = 0$  and  $\beta_2 = 1$ , it follows that  $2w L_w + v L_v = 2w^2 + \alpha v^2 \geq 0$  for all  $\alpha \geq 0$ . In order to find a contact form we need to examine the zero set  $\{p \in M \mid i_{X_H} \lambda(p) = 0\}$  and apply Lemma 3.3.

When  $\alpha > 0$  we have that  $i_X \lambda = 0$  on the set  $S = \{(u, 0, p_u, 0) \mid L(u, 0, 0) = 0\}$ . At any point on  $(u, 0, p_u, 0) \in S$ , with  $p_u \neq 0$ , one has  $p'_v = -p_u \neq 0$  so that all trajectories of the flow immediately leave  $S$  in forward and backward time. If  $p_u = 0$ , then one has

$$p''_v = -p'_u + \alpha v' = -F_u(u) + \alpha p_v = -F_u(u).$$

At  $(u, 0, 0, 0)$  we have  $p''_v = -F_u(u) \neq 0$  and it follows again that orbits of the flow through  $(u, 0, 0, 0)$  leave  $S$  immediately in both time directions. By Lemma 3.3, we see that  $M$  is of contact type, and that one can take  $\lambda + \epsilon df$  to be the contact form, for some  $f \in C^\infty(M)$  and any  $\epsilon \in (0, 1)$ .

If  $\alpha = 0$ , then the set  $S$  is larger,

$$S = \{(u, v, p_u, 0) \mid p_u v = F(u)\}.$$

On  $S$  one has  $p'_v = -p_u$  so that all orbits through points on  $S$  with  $p_u \neq 0$  leave  $S$  in forward and backward time. Points on  $S$  with  $p_u = 0$  are of the form  $(\bar{u}, v, 0, 0)$  with  $F(\bar{u}) = 0$ . Since  $F$  only has simple zeroes, this implies that  $p'_u = F_u(\bar{u}) \neq 0$ , so that trajectories through  $(\bar{u}, v, 0, 0)$  also leave  $S$  in forward and backward time. Again we can apply Lemma 3.3 and conclude that  $M$  is of contact type, concluding the proof of Theorem 1.2.

##### 5. When there is a global Poincaré section

5.1. *Poincaré sections and contact forms.* A surface  $\Sigma \subset M = H^{-1}(0) \subset (\mathbb{R}^4, \omega_0)$  is called a global Poincaré section if (i) it is a closed subset of  $M$ , (ii) the Hamiltonian vector field is transverse to  $\Sigma$ , and (iii) for every  $p \in M$  there exist  $t_- < 0 < t_+$  such that  $\Phi^{t_\pm}(p) \in \Sigma$  (here  $\Phi^t$  denotes the Hamiltonian flow on  $M$  as defined before).

Given a global Poincaré section, one defines the return time  $T : \Sigma \rightarrow \mathbb{R}_+$  by  $T(p) = \inf\{t > 0 \mid \Phi^t(p) \in \Sigma\}$ , and the return map  $\Psi : \Sigma \rightarrow \Sigma$  by  $\Psi(p) = \Phi^{T(p)}(p)$ . Both  $T$  and  $\Psi$  are smooth. The suspension of the return map  $\Psi$  is the space  $\mathcal{B} = (\Sigma \times \mathbb{R})/\sim$ , where the equivalence is given by  $(\Psi(p), t) \sim (p, t + 1)$ . The manifolds  $\mathcal{B}$  and  $M$  are diffeomorphic. The map  $\psi : (p, t) \mapsto \Phi^{tT(p)}(p)$  induces a homeomorphism from  $\mathcal{B}$  to  $M$ . To construct a diffeomorphism one must choose a  $C^\infty$ -function  $\tau : [0, 1] \times (0, \infty) \rightarrow \mathbb{R}$  with  $\partial\tau(t, T)/\partial t > 0$  for all  $(t, T)$ ,  $\partial\tau(t, T)/\partial T = 1$  for  $t$  close to 0 or 1, and  $\tau(0, T) \equiv 0$ ,  $\tau(1, T) \equiv T$ . The map  $\psi : \Sigma \times [0, 1] \rightarrow M$  given by  $\psi(p, t) = \Phi^{\tau(t, T(p))}(p)$  then induces a diffeomorphism from  $\mathcal{B}$  to  $M$ .

LEMMA 5.1. *If  $i_{X_H}(\theta_0) > 0$  outside a compact subset of  $M$  and if  $M$  has a global section, then  $M$  is of contact type<sup>†</sup>.*

*Proof.* The form  $dt$  on  $\mathcal{B}$  is closed (but not exact, since  $t \in \mathbb{R}/\mathbb{Z}$  is multi-valued). Let  $\beta$  be the pushforward of  $dt$  under the diffeomorphism  $\psi : \mathcal{B} \rightarrow M$ . From

$$\frac{\partial\psi}{\partial t} = \frac{\partial\tau}{\partial t} X_H(\Phi^{\tau(t, T(p))}(p))$$

<sup>†</sup> Here  $\theta_0$  is the standard Liouville form with  $d\theta_0 = \omega_0$ . As the referee pointed out, a compact  $M \subset \mathbb{R}^{2n}$  cannot have a global section  $\Sigma$ , otherwise the symplectic form  $\omega = -d\theta_0$  restricted to  $\Sigma$  would be both non-degenerate and exact.

it follows that  $i_{X_H}\beta = \tau_t(t, T(p))^{-1} > 0$  holds at  $\Phi^{\tau(t, T(p))}(p)$ . Since the set where  $i_{X_H}(\theta_0) \leq 0$  is compact,  $i_{X_H}\theta_0$  is bounded from below on  $M$ . For sufficiently large  $k \in \mathbb{R}$  one then has  $i_{X_H}(\theta_0 + k\beta) > 0$  everywhere, so that  $\theta_0 + k\beta$  is a contact form for  $M$ .  $\square$

It is clear from the proof that one can get away with less than compactness of the set  $\{i_{X_H}\theta_0 \leq 0\}$ . Instead, it suffices to verify that

$$\tau_t(t, T(p)) \cdot i_{X_H}(\theta_0)|_{\Phi^{\tau(t, T(p))}(p)} \geq -\delta$$

holds for some positive constant  $\delta$  independent of  $(p, t) \in \Sigma \times [0, 1]$ .

**5.2. Representation by planar Hamiltonian systems.** A planar Hamiltonian system is a system defined by a smooth Hamiltonian  $H(p, q, t)$  on  $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ . Its orbits satisfy the variational problem  $\delta \int p dq - H(p, q, t) dt = 0$ , i.e. they satisfy  $\delta \int \theta = 0$ , where  $\theta$  is the one-form  $\theta = p dq - H dt$ . We continue the discussion of §5.1, assuming from here on that the section  $\Sigma$  is diffeomorphic to  $\mathbb{R}^2$ .

Any isotopy  $\{\Psi^s : \Sigma \rightarrow \Sigma\}_{0 \leq s \leq 1}$  of the identity  $\Psi^0 = \text{id}$  to the return map  $\Psi = \Psi^1$  induces a diffeomorphism of  $\mathcal{B}$  (and, hence,  $M$ ) with  $\Sigma \times (\mathbb{R}/\mathbb{Z})$ . By reparameterizing the  $s$  variable we may assume that  $\Psi^s(p)$  does not depend on  $s$  for  $0 \leq s \leq \frac{1}{3}$  and for  $\frac{2}{3} \leq s \leq 1$ . We can then extend the isotopy  $\Psi^s$  to all  $s \in \mathbb{R}$  by requiring  $\Psi^{s+1} = \Psi \circ \Psi^s$ . The map  $\eta : \Sigma \times \mathbb{R} \rightarrow \Sigma \times \mathbb{R}$  with  $\eta(p, t) = (\Psi^t(p), t)$  then induces a diffeomorphism from  $\Sigma \times (\mathbb{R}/\mathbb{Z})$  to  $\mathcal{B}$ .

Since any orientation preserving diffeomorphism  $\Psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is isotopic to the identity it follows that  $M$  is diffeomorphic with  $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ . (To find an isotopy of  $\Psi$  to the identity first translate so that the origin becomes a fixed point and then consider  $\Psi^s(p) = s^{-1}\Psi(sp)$ ; this is an isotopy of  $\Psi$  to a linear map, namely  $D\Psi(0)$ ; finally, all orientation preserving linear maps are isotopic.)

Because the Hamiltonian vector field is transverse to  $\Sigma$  the two-form  $d\theta_0$  restricted to  $\Sigma$  is nondegenerate.

**PROPOSITION 5.1.** *If  $\int_{\Sigma} d\theta_0$  diverges, then there exist coordinates  $(x, y, t) : M \rightarrow \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ , a smooth function  $K : \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{R}$  and a closed one-form  $\beta$  such that  $\theta_0 = -y dx + K(x, y, t) dt + \beta$ . In particular, the trajectories of the Hamiltonian flow on  $M$  are mapped to those of the planar Hamiltonian system  $\dot{x} = K_y, \dot{y} = -K_x$ .*

*Proof.* Identify  $M$  with  $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ , and assume that we have chosen coordinates  $(u, v)$  on  $\mathbb{R}^2$  so that  $d\theta_0 = f(u, v, t) du \wedge dv$  with  $f(u, v, t) > 0$ . We now construct ‘action angle variables’. Let  $I(u, v, t)$  be the area measured with  $f(u, v, t) du \wedge dv$  of the disc in  $\mathbb{R}^2$  centered at the origin and with radius  $\sqrt{u^2 + v^2}$ . Then  $I$  is a smooth function on  $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ . For each fixed  $t \in \mathbb{R}/\mathbb{Z}$  the orbits of the Hamiltonian vector field of  $I(\cdot, \cdot, t)$  with respect to the symplectic form  $d\theta_0$  are circles centered at the origin. With this choice of  $I$  all orbits have period 1.

Let  $\phi(u, v, t)$  be the time it takes to reach the point  $(u, v, t)$  along the Hamiltonian vector field of  $I(\cdot, \cdot, t)$  starting from the positive  $u$  axis. Then  $\phi(u, v, t)$  is a smooth function on the universal cover of  $(\mathbb{R}^2 \setminus 0) \times (\mathbb{R}/\mathbb{Z})$ , and that  $\phi \bmod \mathbb{Z}$  is smooth on  $(\mathbb{R}^2 \setminus 0) \times (\mathbb{R}/\mathbb{Z})$  itself. By direct computation one verifies that in the coordinates

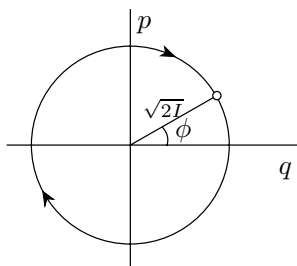


FIGURE 3. Symplectic polar coordinates.

$x = x(u, v, t) = \sqrt{2I} \cos 2\pi\phi$  and  $y = y(u, v, t) = \sqrt{2I} \sin 2\pi\phi$ , the pullback of  $d\theta_0$  to  $\mathbb{R}^2 \times \{t\}$  is  $dx \wedge dy$ . On the entire manifold  $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$  one therefore has  $d\theta_0 = dx \wedge dy + A(x, y, t) dx \wedge dt + B(x, y, t) dy \wedge dt$ .

From  $0 = d d\theta_0 = (B_x - A_y) dx \wedge dy \wedge dt$  it follows that there is a smooth function  $K(x, y, t)$  such that  $A = K_x$  and  $B = K_y$ . Hence  $d\theta_0 = dx \wedge dy + dK \wedge dt$ , and  $\beta = \theta_0 + y dx - K dt$  is closed, as claimed.  $\square$

**5.3. Example of a planar Hamiltonian system which is not of contact type.** As pointed out at the beginning of this section, for non-compact Poincaré sections  $\Sigma$ , a contact form is not immediately found without certain additional requirements.

Consider an autonomous Hamiltonian of the form  $H(p, q, t) = h(I)$  where  $I = \frac{1}{2}(p^2 + q^2)$ , and  $h(I)$  is some smooth increasing function of  $I$ .

The manifold  $M$  on which our flow is defined is  $M = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$  and the Hamiltonian vector field is a pseudo-Reeb vector field for the one-form  $\theta = p dq - h(I) dt$ .

Then the periodic orbits in  $M$  of the Hamiltonian system are exactly those periodic solutions of  $\dot{q} = H_p$ ,  $\dot{p} = -H_q$ , whose minimal period is a fraction  $m/n$  (after  $n$  oscillations the time variable increases by an integer  $m$ , and the  $(q, p)$  variables return to their original positions). In this section we only consider periodic solutions for which  $m = 1$ .

All solutions of the Hamiltonian system are given by

$$p(t) = \sqrt{2I} \sin(\phi - \Omega t), \quad q(t) = \sqrt{2I} \cos(\phi - \Omega t),$$

where the angular frequency is given by  $\Omega = h'(I)$ . A solution has minimal period  $1/n$  if  $h'(I) = 2\pi n$ .

Introduce polar coordinates  $I = (q^2 + p^2)/2$ ,  $\phi = \arctan(p/q)$ . Then along our periodic orbit the angle  $\phi$  decreases by  $2\pi n$ . Using that  $p dq = -I d\phi + d(pq/2)$ , we compute the action of such an orbit:

$$\begin{aligned} A(I) &= \int_{t=0}^1 p dq - h(I) dt \\ &= \int_{t=0}^1 -I d\phi - h(I) dt \\ &= 2n\pi I - h(I) \\ &= h'(I)I - h(I). \end{aligned}$$

If we choose  $h(I) = 4\pi(I - \frac{1}{2} \sin I)$ , then we get

$$A(I) = 2\pi(-I \cos I + \sin I).$$

A sequence of periodic solutions occurs at  $I = k\pi$ ,  $k \in \mathbb{N}$  and their action is  $A(k\pi) = 2k\pi^2(-1)^{k+1}$ . For large  $k$  the action attains arbitrarily large positive and negative values. The Hamiltonian system with  $H(p, q, t) = I + \alpha \sin I$  is therefore not of contact type ( $H^1(M)$  is generated by  $dt$ ).

Although the energy manifold is not of contact type, the Hamiltonian vector field is a Reeb vector field (we are in the special situation that there is a (second) conservation law on  $M$ ). We now construct a contact form  $\mu$  whose Reeb vector field is a multiple of the vector field  $X_H = \partial_t + H_p \partial_q - H_q \partial_p$ . Our Hamiltonian flow on  $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$  is determined by the one form  $\lambda = I d\phi + h(I) dt$ . If there is any other one-form  $\mu$  which determines the same foliation, then for some function  $f(p, q, t)$  one has  $d\lambda = f(p, q, t) d\mu$  (Lemma 2.1). This function  $f$  must be constant on orbits of  $X_H$ , from which it is not hard to see that  $f$  must be a function of  $I$  alone. Thus, we assume  $d\mu = f(I) d\lambda$ , i.e.

$$d\mu = f(I) dI \wedge d\phi + f(I)h'(I) dI \wedge dt = d(J d\phi + g(J) dt)$$

where

$$J(I) = \int_0^I f(\hat{I}) d\hat{I}, \quad g(J) = \int_0^J h'(I(\hat{J})) d\hat{J} + C,$$

where  $I(J)$  is the inverse of  $J(I)$ . We set  $\mu = J d\phi + g(J) dt$ . The form  $\mu$  will be a contact form provided  $\mu \wedge d\mu = -f(Jg'(J) - g(J)) dI \wedge d\phi \wedge dt \neq 0$ , i.e. provided that  $Jg'(J) - g(J) \neq 0$  for all  $J \in \mathbb{R}$ . Using  $g'(J) = h'(I)$  we compute

$$\begin{aligned} Jg'(J) - g(J) &= -C + \int Jg''(J) dJ \\ &= -C + \int J \frac{dI}{dJ} h''(I) dJ \\ &= -C + \int J(I) h''(I) dI \\ &= -C + 2\pi \int J(I) \sin I dI. \end{aligned}$$

Thus, if we let  $J(I) = 1 - e^{-I}$ , and if we choose the constant  $-C$  sufficiently large, then  $\mu$  will be a contact form with the same flow as  $X_H$ , up to reparameterization.

**5.4. Additional remarks.** The example we gave above has an additional integral (namely,  $I$ ) which allows us to write down a large class of one-forms with the same flows. A small non-autonomous perturbation of the Hamiltonian  $H = h(I) + \epsilon h_1(I, \phi, t)$  will, in general, destroy the integral  $I$  and make it impossible to find the form  $\mu$ . It thus seems reasonable to conjecture that an arbitrary small perturbation of  $\lambda$  exists for whose flow is not that of a Reeb vector field on  $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ .

The example is also different from the other classes of Hamiltonian systems we consider in that  $M$  is not embedded as a hypersurface of  $\mathbb{R}^4$ . However, it is easy



to produce such an embedding. Namely, given any strictly positive  $h(I)$ , we consider the Hamiltonian function

$$H(p, q, x, y) = \pi h\left(\frac{1}{2}(p^2 + q^2)\right) - \pi(x^2 + y^2)$$

on  $\mathbb{R}^4$  with symplectic form  $dq \wedge dp + dx \wedge dy$ . This Hamiltonian truly has a second integral, namely  $\pi(x^2 + y^2)$ . Under the Hamiltonian vector field of  $H$  the  $x$  and  $y$  variables undergo harmonic oscillations with period 1.

The zero energy surface  $M = H^{-1}(0)$  is then diffeomorphic to  $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$  via  $(p, q, t) \mapsto (p, q, R \cos 2\pi t, R \sin 2\pi t)$  with  $R = \sqrt{\frac{1}{2}h(p^2 + q^2)}$ . On  $M$  one has  $\frac{1}{2}(y dx - x dy) = -R^2 dt$ , so that

$$p dq + \frac{1}{2}(y dx - x dy) = p dq - h(I) dt,$$

as required.

#### 6. Poincaré sections for second-order Lagrangians

Following [3] we can define a transverse slice to the flow on  $M$  provided that the Lagrangian  $L(u, v, w)$  is a strictly convex function of  $w$  which satisfies

$$\inf_{w \in \mathbb{R}} L(u, 0, w) > 0$$

for all  $u \in \mathbb{R}$ , and

$$\lim_{w \rightarrow \pm\infty} \frac{L(u, 0, w)}{w} = \pm\infty.$$

Under this hypothesis the Legendre transform  $L^*(u, 0, p_v) = \sup_w \{p_v w - L(u, 0, w)\}$  satisfies  $L^*(u, 0, 0) = -\inf_w L(u, 0, w) < 0$ . Since  $L^*(u, 0, p_v)$  is a strictly convex and proper function of  $p_v$ , there are, for each  $u \in \mathbb{R}$ , precisely two solutions  $p_v^-(u) < 0 < p_v^+(u)$  of  $L^*(u, 0, p_v) = 0$ . At the positive solution one has  $\partial L^*/\partial p_v > 0$ .

On  $M$  the equation  $v = 0$  implies  $L^*(u, 0, p_v) = 0$ . Hence, the set  $M \cap \{v = 0\}$  consists of two smooth surfaces

$$\Sigma^\pm = \{(u, 0, p_u, p_v) \mid u, p_u \in \mathbb{R}, p_v = p_v^\pm(u)\}.$$

Both  $\Sigma^\pm$  are transverse slices for the flow since, e.g., on  $\Sigma^+$  one has  $\dot{v} = \partial L^*/\partial p_v > 0$ .

To verify that  $\Sigma^+$  is a Poincaré section we must show that all orbits return to  $\Sigma^+$  in both time directions. We do this in the special case where  $L(u, v, w) = \frac{1}{2}w^2 + (\alpha/2)v^2 + F(u)$ .

6.1. *Proof of Theorem 1.3.* Assume that  $F(u) > 0$ . Then the section  $\Sigma^+$  is given by

$$\Sigma = \{(u, 0, p_u, \sqrt{2F(u)}) \mid u, p_u \in \mathbb{R}\}.$$

If we also assume that for large  $u$  the potential  $F(u)$  grows superquadratically (to be precise,  $F(u)/u^2 \rightarrow \infty$  as  $u \rightarrow \pm\infty$ ) all solutions oscillate, i.e. for any solution  $u(t)$  of the Euler–Lagrange–Poisson equations and any  $t_0$  there exist  $t_- < t_0 < t_+$  at which the solution has local minima. Thus, any orbit of the Hamiltonian flow on  $M$  returns to  $\Sigma$  both in forward and backward time. This has been shown in [16] for  $F(u) = \frac{1}{4}(1 - u^2)^2$  and with a minor modification the proof can be generalized to superquadratic potentials.

So  $\Sigma$  is a global Poincaré section for  $M$  and clearly  $\Sigma$  is diffeomorphic with the plane, while  $\int_{\Sigma} d\theta_0 = \infty$ . Therefore, the Hamiltonian flow on  $M$  is conjugate to a planar Hamiltonian system, as was explained in Proposition 5.1.

Assuming also that  $\alpha < 0$  we can find an explicit diffeomorphism from  $M$  to a planar Hamiltonian system. Write  $\alpha = -a^2$  and define

$$I = \frac{1}{2}(av^2 + p_v^2/a), \quad \varphi = \arctan \frac{p_v}{av},$$

i.e.

$$p_v = \sqrt{2aI} \sin \varphi, \quad v = \sqrt{2I/a} \cos \varphi.$$

Thus,  $\varphi$  is a smooth function on the universal cover of  $M$ . A short calculation reveals that

$$-I d\varphi = p_v dv - d\left(\frac{vp_v}{2}\right)$$

so that the canonical form on  $M$  is given by  $\theta_0 = p_u du - I d\varphi + d(vp_v/2)$ . The Hamiltonian as a function of  $(u, p_u, I, \varphi)$  is  $H = aI + p_u \sqrt{2I/a} \cos \varphi - F(u)$ . On its zeroset, one therefore has

$$I = \left\{ -\frac{p_u}{\sqrt{2a^3}} \cos \varphi + \sqrt{\frac{p_u^2}{2a^3} \cos^2 \varphi + \frac{F(u)}{a}} \right\}^2. \quad (9)$$

The trajectories of the Hamiltonian vector field on  $M$  are determined by the ‘principle of least action’ [2, §45C]  $\delta \int \theta_0 = 0$ , i.e. by  $\delta \int p_u du - I(u, p_u, \varphi) d\varphi = 0$ , where we regard  $I$  as the smooth function of  $(u, p_u, \varphi)$  specified in (9). Consequently they are integral curves of the Hamilton equations of  $I(u, p_u, \varphi)$ , i.e.

$$\frac{du}{d\varphi} = \frac{\partial I(u, p_u, \varphi)}{\partial p_u}, \quad \frac{dp_u}{d\varphi} = -\frac{\partial I(u, p_u, \varphi)}{\partial u}.$$

**6.2. Proof of Theorem 1.4.** We claim that for some  $M < \infty$  any bounded solution  $u(t)$  of the Euler–Poisson equations is actually bounded by  $|u(t)| \leq M$ . This then immediately implies that  $|u'''' - \alpha u''| \leq M'$  and, hence, using an interpolation inequality, that all derivatives  $u^{(j)}$  of order  $j \leq 4$  are also uniformly bounded. It follows that the Hamiltonian flow on the level set  $M$  has a largest compact invariant set  $K \subset M$ .

Although this may be well known to some, let us prove our claim. Multiply the equation with  $u(t)$  and  $h(t) = (1 + (t - a)^2)^{-1}$  ( $a \in \mathbb{R}$ ), and integrate by parts to get

$$\int_{\mathbb{R}} [h(uF'(u) + (u'')^2) + (2h'' - \alpha h)uu'' - \frac{1}{2}h''''u^2] dt = 0.$$

Use  $|h''|, |h''''| \leq Ch$  and also  $uu'' \leq \frac{1}{\epsilon}u^2 + \epsilon(u'')^2$  to obtain

$$\int_{\mathbb{R}} h[uF'(u) - C'u^2 + \frac{1}{2}(u'')^2] dt \leq 0.$$

The superlinearity of  $F'(u)$  implies  $uF'(u) - C'u^2 \geq u^2 - C''$ . One gets  $\int h(u^2 + (u'')^2) dt \leq C'''$  independent of  $u$  and  $a$ . This leads to the asserted  $L^\infty$ -bound.

Using the Poincaré section for the flow as in the proof of Lemma 5.1, we obtain a closed one-form  $\beta$  on  $M$  with  $i_X\beta > 0$  everywhere. Since  $K$  is compact  $i_X\theta_0$  is bounded from below on  $K$  by some positive constant. We can therefore choose  $k > 0$  large enough so that  $i_X(\theta_0 + k\beta) > 0$  on  $K$ . Lemma 3.4 then provides us with a smooth function  $f$  on  $M$  for which  $\theta_0 + k\beta + df$  is a contact form.

### 7. Topology of $M$

Instead of analysing the topology of energy manifolds of second-order Lagrangians in general, we restrict ourselves here to the Swift–Hohenberg and eFK Lagrangians. In this case, an energy manifold is given by the equation

$$H = \frac{1}{2}p_v^2 + p_u v - \frac{\alpha}{2}v^2 - F(u) = 0.$$

By rewriting this as

$$H = \frac{1}{2}p_v^2 + \frac{1}{4}\{(p_u + (1 - \alpha/2)v)^2 - (p_u - (1 + \alpha/2)v)^2\} - F(u) = 0$$

and introducing the new coordinates

$$X = \frac{1}{2}\{p_u + (1 - \alpha/2)v\} \quad \text{and} \quad Y = \frac{1}{2}\{p_u - (1 + \alpha/2)v\},$$

we see that  $M$  is given by

$$\frac{1}{2}p_v^2 + X^2 = Y^2 + F(u). \quad (10)$$

It is immediately clear that all hypersurfaces obtained by varying  $\alpha \in \mathbb{R}$  are diffeomorphic.

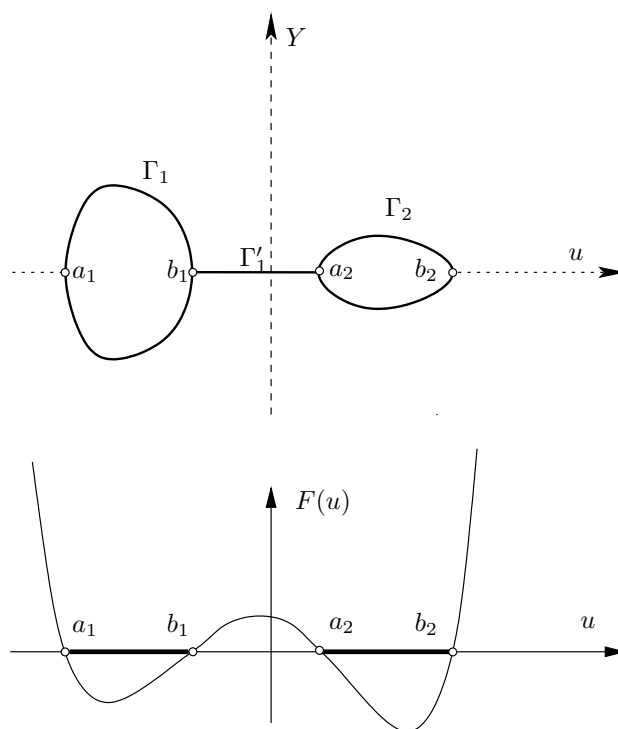
If  $F(u) \geq c > 0$  for some constant  $c$ , then (10) also shows that  $M$  is diffeomorphic to  $S^1 \times \mathbb{R}^2$ , since one can parameterize  $M$  by  $p_v = \sqrt{2Y^2 + 2F(u)} \cos \theta$ ,  $Y = \sqrt{Y^2 + F(u)} \sin \theta$ , where  $Y, u \in \mathbb{R}$  and  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$  are the parameters.

We can also easily compute the homotopy type of  $M$  for general  $F$ . Let us assume that  $F(u) > 0$  for large enough  $|u|$ , and that  $F$  only has simple zeroes. Then the set  $\{u \in \mathbb{R} \mid F(u) \leq 0\}$  is the union of a finite number of intervals, say  $[a_1, b_1] \cup \dots \cup [a_k, b_k]$ . The set in the  $(u, Y)$ -plane where  $Y^2 + F(u)$  vanishes is the union of  $k$  closed curves  $\Gamma_j$ , given by  $Y = \pm\sqrt{F(u)}$ , with  $u \in [a_j, b_j]$ ,  $1 \leq j \leq k$ . See Figures 4 and 5.

Let  $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  be the projection in  $(u, X, Y, p_v)$  space onto the  $(u, Y)$  plane. Then  $\pi(M)$  is the region outside the curves  $\Gamma_j$ , with the curves  $\Gamma_j$  included. Over each  $(u, Y)$  in the interior of  $\pi(M)$  the preimage  $\pi^{-1}(u, Y)$  is a circle whose radius shrinks to zero as the point  $(u, Y)$  approaches one of the curves  $\Gamma_j$ .

One can deform the projection  $\pi(M)$  onto the union of the circles  $\Gamma_j$  and the  $k - 1$  line segments  $\Gamma'_j = \{(u, 0) \mid b_j \leq a_{j+1}\}$ . This deformation can be lifted to a deformation retraction of  $M$  onto  $\pi^{-1}(\Gamma_1 \cup \dots \cup \Gamma_k \cup \Gamma'_1 \cup \dots \cup \Gamma'_{k-1})$ . Since each  $\pi^{-1}(\Gamma_j)$  is an  $S^1$  and each  $\pi^{-1}(\Gamma'_j)$  is an  $S^2$  one finds that  $M$  has the homotopy type of a bouquet of  $k$  circles and  $k - 1$  two-spheres. The first singular homology group is generated by the  $\pi^{-1}\Gamma_j$ , the second homology group is generated by  $\pi^{-1}(\Gamma'_j)$ .

For instance, when  $k = 1$ , one finds that  $M$  has the homotopy type of a circle. In this case one actually finds that  $M$  is diffeomorphic with  $S^1 \times \mathbb{R}^2$ .

FIGURE 4. The  $(u, Y)$  plane.

If one applies a similar analysis to the case in which  $F(u) > 0$  in some bounded interval  $(b_1, a_2)$  and  $F(u) < 0$  on  $(-\infty, b_1) \cup (a_2, \infty)$ , then one finds that  $M$  has the homotopy type of  $S^2$  and, in particular, is simply connected. One finds that  $M$  is diffeomorphic with  $\mathbb{R} \times S^2$ .

#### 8. $F(u)$ arbitrary, $\alpha \ll 0$

8.1. *A very singular perturbation problem.* In this section we present the example promised in Theorem 1.5. Consider the Swift–Hohenberg Lagrangian

$$L(u, v, w) = \frac{1}{2}w^2 + \frac{\alpha}{2}v^2 + F(u), \quad \alpha < 0.$$

Rescale time by  $t \rightarrow \sqrt{\epsilon}t$ , and set  $\alpha = -1/\epsilon$ . This yields a more convenient formulation. The Lagrangian now becomes

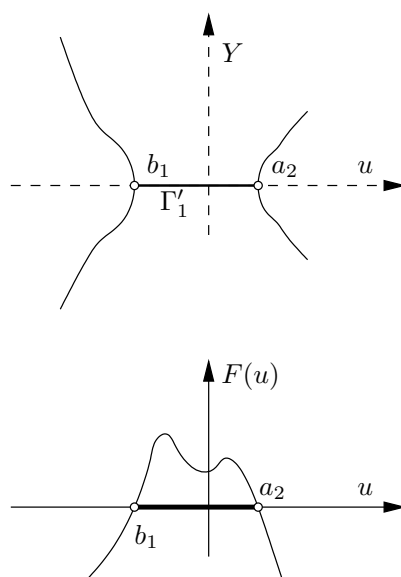
$$L(u, v, w) = \frac{\epsilon^2}{2}w^2 - \frac{1}{2}v^2 + F(u) \quad (11)$$

in which  $\epsilon$  is a small positive constant. The variational equation for this Lagrangian is

$$\epsilon^2 u'''' + u'' + F_u(u) = 0. \quad (12)$$

As  $\epsilon \searrow 0$ , (12) formally reduces to a second-order equation

$$u'' + F_u(u) = 0. \quad (13)$$

FIGURE 5. The case  $M \cong S^2 \times \mathbb{R}$ .

We will verify that for small enough  $\epsilon$  solutions of (12) are essentially solutions of the second-order equation (13), with a small rapid oscillation superimposed. This will allow us to construct many periodic orbits on the corresponding zero-energy manifold  $M$  and compute their actions.

We again pass to the Hamiltonian formulation of (12) and introduce new variables  $p_u$  and  $p_v$ , where now

$$p_v = \frac{\partial L}{\partial w} = \epsilon^2 w.$$

Thus,  $u$  is a solution of (12) if and only if  $(u, v, p_u, p_v)$  is a solution of the Hamiltonian system with Hamiltonian function

$$H(u, v, p_u, p_v) = \frac{p_v^2}{2\epsilon^2} + p_u v + \frac{1}{2} v^2 - F(u).$$

We rewrite this as

$$H = \frac{p_v^2}{2\epsilon^2} + \frac{1}{2} (v + p_u)^2 - \left\{ \frac{1}{2} p_u^2 + F(u) \right\},$$

and apply the following coordinate change

$$U = u + p_v, \quad p_U = p_u, \quad V = \sqrt{\epsilon}(v + p_u), \quad p_V = \frac{p_v}{\sqrt{\epsilon}}. \quad (14)$$

It follows from  $p_U dU + p_V dV = p_u du + p_v dv + d(p_u p_v)$  that this is a symplectic coordinate change.

The Hamiltonian in these new coordinates is

$$H = \frac{1}{2\epsilon} (p_V^2 + V^2) - \left\{ \frac{1}{2} p_U^2 + F(U - \sqrt{\epsilon} p_V) \right\}.$$

The Hamiltonian equations are

$$\begin{aligned}\dot{U} &= -p_U, & \dot{p}_U &= F'(U - \sqrt{\epsilon} p_V), \\ \dot{V} &= \epsilon^{-1} p_V + \sqrt{\epsilon} F'(U - \sqrt{\epsilon} p_V), & \dot{p}_V &= -\epsilon^{-1} V.\end{aligned}\quad (15)$$

We will consider solutions which lie on the zero energy manifold  $M = H^{-1}(0)$ . In particular, we will assume  $\epsilon$  is small and that both sides in the identity

$$\frac{1}{2\epsilon}(p_V^2 + V^2) = \frac{1}{2}p_U^2 + F(U - \sqrt{\epsilon} p_V) \quad (16)$$

which defines  $M$  are bounded. Then  $V$  and  $p_V$  are of order  $\sqrt{\epsilon}$ , and the Hamiltonian equations are approximately given by two uncoupled two-dimensional systems,

$$\dot{U} = -p_U, \quad \dot{p}_U = F'(U) \quad (17)$$

and

$$\dot{V} = \epsilon^{-1} p_V, \quad \dot{p}_V = -\epsilon^{-1} V. \quad (18)$$

The first of these is the Hamiltonian system corresponding to the second order ODE (13). The second is a simple harmonic oscillator with angular frequency  $\epsilon^{-1}$ . If we assume that  $(V, p_V)$  is given by

$$V = \sqrt{2I\epsilon} \sin(t/\epsilon - \phi), \quad p_V = \sqrt{2I\epsilon} \cos(t/\epsilon - \phi),$$

then the zero total energy condition (16) forces  $(U, p_U)$  to be a solution of (17) with

$$\frac{1}{2}p_U^2 + F(U) = I. \quad (19)$$

Since  $\sqrt{2I\epsilon}$  is the amplitude of the  $(V, p_V)$  oscillation we must always have  $I \geq 0$ . Formally, one would expect solutions of the Swift–Hohenberg equation (12) to be approximated by

$$u(t) = U(t) - \epsilon\sqrt{2I} \cos(t/\epsilon - \gamma) + o(\epsilon), \quad (20)$$

where  $(U(t), p_U(t))$  is a solution of (17) with energy  $I \geq 0$ , and  $\gamma \in [0, 2\pi)$  is some phase angle.

**8.2. Action angle variables.** We replace  $(V, p_V)$  by new coordinates  $(I, \phi)$  given by

$$I = \frac{1}{2}(p_V^2 + V^2), \quad \phi = \arctan \frac{p_V}{V}$$

and thus  $V = \sqrt{2I} \cos \phi$ ,  $p_V = \sqrt{2I} \sin \phi$ . One has  $I d\phi = \frac{1}{2}(V dp_V - p_V dV) = -p_V dV + \frac{1}{2} d(V p_V)$ , so that

$$p_U dU + p_V dV = p_U dU - I d\phi + \frac{1}{2} d(V p_V). \quad (21)$$

It follows that  $dU \wedge dp_U + dV \wedge dp_V = dU \wedge dp_U + dI \wedge d\phi$ , so that  $(U, p_U, I, \phi)$  are symplectic coordinates. The Hamiltonian  $H$  appears in these variables as

$$H = \frac{I}{\epsilon} - \left\{ \frac{1}{2}p_U^2 + F(U - \sqrt{2I\epsilon} \cos \phi) \right\}.$$

Define

$$\Omega_\epsilon = \{(U, p_U) \in \mathbb{R}^2 \mid c \leq \frac{1}{2}p_U^2 + F(U) \leq C, |U| + |p_U| \leq \bar{C}\}$$

for certain  $0 < c < C$ ,  $\bar{C} < \infty$  with  $c$  small and  $C$  and  $\bar{C}$  large. We also define  $M_\epsilon$  to be the portion of  $M$  for which  $(U, p_U)$  lies in  $\Omega_\epsilon$ . After rewriting the equation  $H = 0$  as

$$\frac{I}{\epsilon} = \frac{1}{2}p_U^2 + F\left(U - \epsilon\sqrt{2\frac{I}{\epsilon}}\cos\phi\right),$$

one concludes from the implicit function theorem that for small enough  $\epsilon$  there is a unique solution  $I = \mathcal{I}(U, p_U, \phi, \epsilon)$  which is smooth in  $\epsilon$  and whose Taylor series begins with

$$\mathcal{I}(U, p_U, \phi, \epsilon) = \epsilon I_0(U, p_U) + \epsilon^2 I_1(U, p_U, \phi) + \mathcal{O}(\epsilon^3), \quad (22)$$

where  $I_0 = \frac{1}{2}p_U^2 + F(U)$ ,  $I_1 = -F'(U)\sqrt{p_U^2 + 2F(U)}\cos\phi$ . In particular, we see that for small  $\epsilon$  the portion  $M_\epsilon$  of the zero energy manifold is diffeomorphic with the product space  $\Omega_\epsilon \times S^1$ .

On  $M_\epsilon$  we have

$$\frac{d\phi}{dt} = -\frac{\partial H}{\partial I} = -\frac{1}{\epsilon} - F'(U - \sqrt{2I\epsilon}\cos\phi)\sqrt{\frac{\epsilon}{2I}}\cos\phi,$$

while for small enough  $\epsilon$  one has  $c' < I/\epsilon < C'$  for any  $0 < c' < c < C < C' < \infty$ . Therefore, for small  $\epsilon$  one has

$$\frac{d\phi}{dt} = -\frac{1}{\epsilon} + \mathcal{O}(1) < 0,$$

so we may parameterize orbits of  $X_H$  by the angle variable  $\phi$  instead of time  $t$ . Orbits are then determined by specifying  $(U, p_U)$  as functions of  $\phi$ ; given the  $(U, p_U)$  component of an orbit, one recovers the action from  $I = \mathcal{I}(U, p_U, \phi; \epsilon)$ .

On  $M_\epsilon$ , orbits satisfy the least action principle, i.e.

$$\delta \int p_U dU + p_V dV = 0, \quad \text{i.e. } \delta \int p_U dU - \mathcal{I}(U, p_U, \phi, \epsilon) d\phi = 0,$$

where we have used (21). Hence,  $(U, p_U)$  as function of  $\phi$  satisfies the Hamiltonian equations with Hamiltonian  $\mathcal{I}(U, p_U, \phi, \epsilon)$ . Moreover, (21) shows us that if  $(U(\phi), p_U(\phi))$  is a  $2N\pi$  periodic solution of

$$\frac{dU}{d\phi} = \frac{\partial \mathcal{I}}{\partial p_U}, \quad \frac{dp_U}{d\phi} = -\frac{\partial \mathcal{I}}{\partial U}, \quad (23)$$

then the corresponding periodic solution of (15) has action

$$\mathcal{A} = \oint p_U dU + p_V dV = \int_0^{2N\pi} p_U dU - \mathcal{I}(U, p_U, \phi, \epsilon) d\phi.$$

We write (23) out to get

$$\begin{aligned} \frac{1}{\epsilon} \frac{dU}{d\phi} &= p_U + \epsilon \frac{\partial I_1(U, p_U, \phi)}{\partial p_U} + \mathcal{O}(\epsilon^2), \\ \frac{1}{\epsilon} \frac{dp_U}{d\phi} &= -F'(U) - \epsilon \frac{\partial I_1(U, p_U, \phi)}{\partial U} + \mathcal{O}(\epsilon^2). \end{aligned}$$

Introducing a new time variable  $\tau = \epsilon\phi$  we get

$$\begin{aligned}\frac{dU}{d\tau} &= p_U + \epsilon \frac{\partial I_1(U, p_U, \tau/\epsilon)}{\partial p_U} + \mathcal{O}(\epsilon^2), \\ \frac{dp_U}{d\tau} &= -F'(U) - \epsilon \frac{\partial I_1(U, p_U, \tau/\epsilon)}{\partial U} + \mathcal{O}(\epsilon^2)\end{aligned}\quad (24)$$

where the  $\mathcal{O}(\epsilon^2)$  terms are smooth functions of  $(U, p_U, \tau/\epsilon)$  and all terms are  $2\pi$ -periodic in  $\tau/\epsilon = \phi$ .

Denote the flow determined by these ODEs by  $\Phi_\epsilon^\tau$ . Since the perturbation terms in (24) are uniformly smooth in  $(U, p_U)$  the flow  $\Phi_\epsilon^\tau$  is an  $\mathcal{O}(\epsilon)$  perturbation of  $\Phi_\epsilon^\tau$  in the  $C^\infty$  topology<sup>†</sup>. Here  $\Phi_0^\tau$  is the flow corresponding to the autonomous equations

$$\frac{dU}{d\tau} = p_U, \quad \frac{dp_U}{d\tau} = -F'(U). \quad (25)$$

These are the Hamiltonian equations for  $H_0 = \frac{1}{2}p_U^2 + F(U)$ , and thus periodic orbits of (25) come in families parameterized by their energy  $E = H_0(U, p_U)$ . On any such family, the period  $T$  depends smoothly on the energy  $E$ . We shall call a periodic orbit *non-degenerate* if at this orbit one has  $dT/dE \neq 0$ .

Assume that  $(\bar{U}(\tau), \bar{p}_U(\tau))$  is a  $T$  periodic non-degenerate solution of (25). It traces out a closed curve  $\gamma$  in the  $(U, p_U)$  plane. This curve consists of fixed points for the map  $\Phi_0^T$ . The non-degeneracy condition implies that for small  $\epsilon > 0$ , the map  $\Phi_\epsilon^T$  will have at least one fixed point in an  $\epsilon$  neighborhood of  $\gamma$ . In general, there will be many fixed points. If  $T = 2N\pi\epsilon$  for some  $N \in \mathbb{N}$  then these fixed points correspond to periodic solutions of (24) which are  $\epsilon$  close to  $(\bar{U}(\tau - \tau_0), \bar{p}_U(\tau - \tau_0))$  for some phase  $\tau_0$ . The action of this solution is

$$\begin{aligned}\mathcal{A} &= \int p_U dU - \mathcal{I}(U, p_U, \phi; \epsilon) d\phi \\ &= \int p_U dU - \frac{\mathcal{I}}{\epsilon} d\tau.\end{aligned}$$

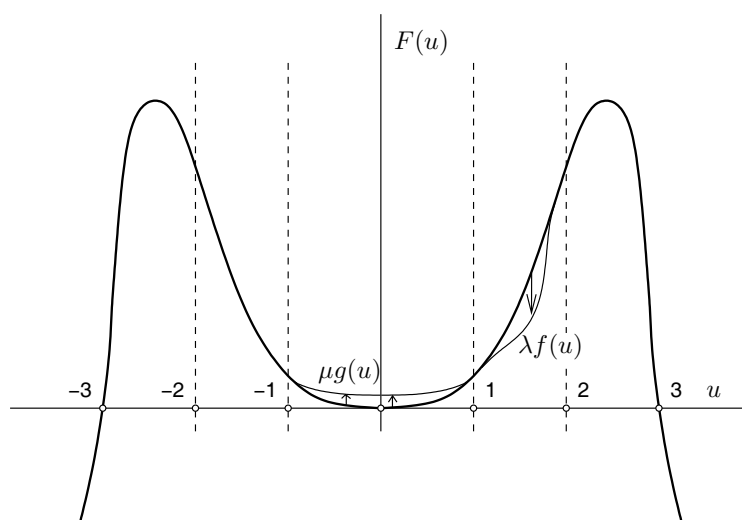
Now use (24) and (22) to replace  $dU/d\tau$  and  $\mathcal{I}/\epsilon$ , respectively. One obtains

$$\begin{aligned}\mathcal{A} &= \int_0^T \left\{ \frac{1}{2} \bar{p}_U^2 - F(\bar{U}) + \mathcal{O}(\epsilon) \right\} d\tau \\ &= \int_0^T \left\{ \frac{1}{2} \bar{p}_U^2 - F(\bar{U}) \right\} d\tau + \mathcal{O}(\epsilon).\end{aligned}$$

Thus, we have proved that near any non-degenerate  $T = 2\pi N\epsilon$  periodic orbit  $(\bar{U}, \bar{p}_U)$  of (25) there is a  $T$  periodic solution of (24) whose action is within  $\mathcal{O}(\epsilon)$  equal to the Lagrangian action of  $(\bar{U}, \bar{p}_U)$ .

<sup>†</sup> In fact, by the method of single frequency averaging [2, §52] the return map  $\Phi_\epsilon^\tau$  with  $\tau = 2N\pi\epsilon$ ,  $N \in \mathbb{N}$ , is an  $\mathcal{O}(\epsilon^2)$  perturbation of the return map of the averaged Hamiltonian. Since the first-order term in (22) has zero time average,  $\Phi_\epsilon^\tau$  is  $\mathcal{O}(\epsilon^2)$  close to  $\Phi_0^\tau$  in the  $C^\infty$  topology.



FIGURE 6. The potential  $F(u)$ .

8.3. *Simple mechanical systems.* Consider the mechanical system with Lagrangian  $L(u, \dot{u}) = \frac{1}{2}\dot{u}^2 - F(u)$ . The energy of a periodic orbit is  $E = \frac{1}{2}\dot{u}^2 + F(u)$ . Using  $dt = du/\dot{u}$  one finds that the action of such an orbit is

$$S = \oint L dt = \oint \frac{\frac{1}{2}\dot{u}^2 - F(u)}{\sqrt{2(E - F(u))}} du = \oint \frac{E - 2F(u)}{\sqrt{2(E - F(u))}} du.$$

For the special case of the harmonic oscillator,  $L = \frac{1}{2}(\dot{u}^2 - a^2 u^2)$  all periodic orbits are of the form  $u(t) = A \sin a(t - t_0)$ , and by direct substitution one finds

$$S = \int_0^{2\pi/a} \frac{1}{2} (A^2 a^2 \cos^2 a(t - t_0) - A^2 a^2 \sin^2 a(t - t_0)) dt = 0.$$

In other words, *the action of any orbit of the harmonic oscillator vanishes.*

LEMMA 8.1. *There exists a potential  $F : \mathbb{R} \rightarrow \mathbb{R}$  with  $F(u) > 0$  on  $-L < u < L$  and  $F(u) < 0$  when  $|u| > L$  such that  $\ddot{u} + F_u(u) = 0$  has one periodic orbit with positive Lagrangian action and one with negative Lagrangian action.*

*Proof.* We construct the potential by perturbing  $F(u) = \frac{1}{2}u^2$ . Let  $0 \leq f \in C^\infty(\mathbb{R})$  be supported in  $1 < u < 2$ . Then, for small  $\lambda$ , the potential  $F_\lambda(u) = F(u) - \lambda f(u)$  has a periodic orbit  $u_\lambda(t)$  which oscillates between  $-2$  and  $2$  and, hence, has energy  $E = 2$ . The action  $S_\lambda$  of this orbit satisfies

$$\left. \frac{dS_\lambda}{d\lambda} \right|_{\lambda=0} = \sqrt{2} \int_{-2}^2 \frac{\frac{3}{2}E - F(u)}{(E - F(u))^{3/2}} f(u) du = 2 \int_{-2}^2 \frac{6 - u^2}{(4 - u^2)^{3/2}} f(u) du > 0.$$

For sufficiently small  $\lambda > 0$  the amplitude 2 orbit of the potential  $F_\lambda$  will have positive action. Since  $F_\lambda$  coincides with the quadratic potential  $u^2/2$  in the interval  $|u| \leq 1$ , the potential  $F_\lambda$  still has an amplitude 1 orbit ( $u(t) = \cos t$ ) with zero action. We now perturb  $F_\lambda$  to  $F_{\lambda,\mu} = F_\lambda + \mu g(u)$ , where  $0 \leq g \in C^\infty(\mathbb{R})$  is supported in  $|u| < 1$ . See Figure 6.

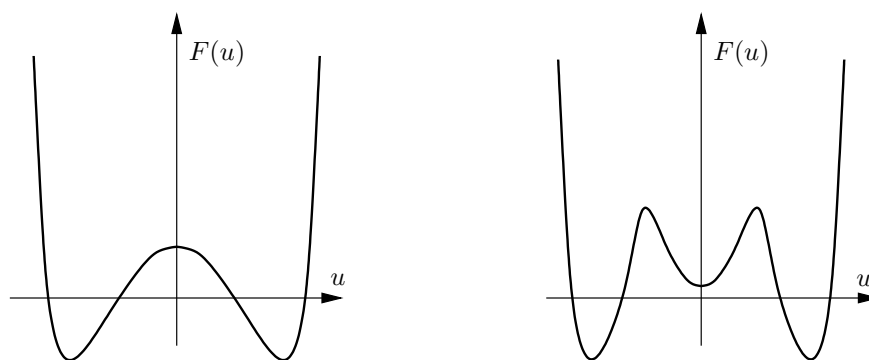


FIGURE 7. Swift-Hohenberg and modified Swift-Hohenberg potentials.

Reasoning as above, we find that for sufficiently small  $\mu > 0$  the amplitude 1 orbit of the potential  $F_{\lambda,\mu}$  will have negative action. By first choosing  $\lambda > 0$  small but fixed, so that the amplitude 2 orbit has positive action, and then choosing  $\mu > 0$  small enough we can guarantee that the amplitude 2 orbit of  $F_{\lambda,\mu}$  still has positive action, while the amplitude 1 orbit has negative action.

Since the amplitude 1 and 2 orbits are unaffected by changes in the potential outside the interval  $|u| \leq 2$ , we may define  $F(u)$  as we like for  $|u| > 2$  and, in particular, we could make it vanish at  $u = \pm 3$  and also be negative for  $|u| > 3$ .  $\square$

8.4. *The example.* We choose  $F(u)$  as in Lemma 8.1. Then for sufficiently small  $\epsilon > 0$  the zero-energy manifold  $M = H^{-1}(0)$ ,

$$H = \frac{p_v^2}{\epsilon} + p_u v + \frac{1}{2}v^2 - F(u)$$

contains two periodic orbits with actions of opposite signs. Furthermore, since  $F(u)$  is negative outside of the interval  $|u| \leq 3$ , the manifold  $M$  is homeomorphic to  $S^2 \times \mathbb{R}$ , i.e.  $M$  is simply connected. By Lemma 3.2,  $M$  cannot be of contact type.

A different example for a potential of globally the same shape could be found via the methods used in [3], where simple closed periodic orbits are found with estimates on their actions. This also yields periodic solutions with both negative and positive action, but with an explicit estimate for the range of the parameter  $\alpha$  for which  $M_\alpha$  is not of contact type.

8.5. *More examples.* It is very easy to extend the example in §8.4 to other examples. Namely, one can modify the potential  $F(u)$  outside the interval  $-3 \leq u \leq 3$  any way one likes, and the resulting hypersurface  $M$  will still not be of contact type. Indeed, the new  $M$  contains the two periodic orbits with opposite actions, and they are still contractible (since the contraction takes place in the region  $M \cap \{|u| \leq 3\}$ ).

Thus, the potential on the right in Figure 7 contains our previous example and hence yields an energy manifold  $M$  which is not of contact type. It can be deformed into the potential  $F(u) = \frac{1}{4}(1 - u^2)^2 + E$  (with  $-\frac{1}{4} < E < 0$ ) of the Swift-Hohenberg

equation without changing the number of zeros of the potential  $F$ , and without changing the topology of the hypersurface  $M$ . As we observed in the introduction, we do not know whether  $M$  with the Swift–Hohenberg potential is of contact type or not.

*Acknowledgements.* The third author wishes to thank R. W. Ghrist for the stimulating discussions and encouragement on this subject. The second and third authors are supported by NWO-Veni/Vidi grants 639.031.204 and 639.032.202, respectively. The first author was supported by NSF on grant DMS-0101124 and a research visiting grant of NWO with additional support from the NDNS+ NWO research cluster and the Department of Mathematics of the Vrije Universiteit Amsterdam. We would also like to thank H. Hofer, V. Ginzburg and the referee for providing us with several references and useful suggestions.

## REFERENCES

- [1] V. I. Arnol'd and S. P. Novikov (Eds.). *Dynamical Systems IV: Symplectic Geometry and its Applications (Encyclopedia of Mathematics, 4)*. Springer, Berlin, 1990.
- [2] V. I. Arnol'd. *Mathematical Methods of Classical Mechanics (Graduate Texts in Mathematics, 60)*. Springer, New York, 1978.
- [3] J. B. van den Berg and R. C. Vandervorst. Second order Lagrangian twist systems: simple closed characteristics. *Trans. Amer. Math. Soc.* **354** (2002), 1393–1420.
- [4] J. B. van den Berg. Dynamics and equilibria of fourth order differential equations. *PhD Thesis*, University of Leiden, December 2000.
- [5] K. Cieliebak. A geometric obstruction to the contact type property. *Math. Z.* **228** (1998), 451–487.
- [6] V. L. Ginzburg. An embedding  $S^{2n-1} \rightarrow \mathbb{R}^{2n}$ ,  $2n - 1 \geq 7$ , whose Hamiltonian flow has no periodic trajectories. *Internat. Math. Res. Notices* **2** (1995), 83–97.
- [7] V. L. Ginzburg. Hamiltonian dynamical systems without periodic orbits. *Proc. Northern California Symplectic Geometry Seminar (American Mathematical Society Translation Series (2), 196)*. American Mathematical Society, Providence, RI, 1999, pp. 35–48.
- [8] V. Ginzburg and B. Gürel. A  $C^2$ -smooth counterexample to the Hamiltonian Seifert conjecture in  $\mathbb{R}^4$ . *Ann. of Math.* **158** (2003), 953–976.
- [9] R. W. Ghrist, J. B. Van den Berg and R. C. Vandervorst. Morse theory on spaces of braids and Lagrangian dynamics. *Invent. Math.* **152**(2) (2003), 369–432.
- [10] M. R. Herman. Examples of compact hypersurfaces in  $\mathbb{R}^{2p}$ ,  $2p \geq 6$ , with no periodic orbits. *Proc. Hamiltonian Systems with Three or More Degrees of Freedom (S'Agaró, 1995) (NATO Advanced Scientific Instruments Series C Mathematical and Physical Sciences, 533)*. Kluwer, Dordrecht, 1999, p. 126.
- [11] H. Hofer and E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics*. Birkhäuser, Basel, 1994.
- [12] H. Hofer and E. Zehnder. Pseudoholomorphic curves and dynamics. *Proc. Arnoldfest (Toronto, ON, 1997) (Fields Institute Communications, 24)*. American Mathematical Society, Providence, RI, 1999, pp. 225–239.
- [13] J. Hulshof, J. B. van den Berg and R. C. Vandervorst. Traveling waves for fourth order parabolic equations. *SIAM J. Math. Anal.* **32** (2001), 1342–1374.
- [14] F. Laudenbach and J.-C. Sikorav. Hamiltonian disjunction and limits of Lagrangian submanifolds. *Internat. Math. Res. Notices* **4** (1994), 161–168.
- [15] D. McDuff. Applications of convex integration to symplectic and contact geometry. *Ann. Inst. Fourier* **37** (1987), 107–133.
- [16] L. A. Peletier and W. C. Troy. Chaotic spatial patterns described by the extended Fisher Kolmogorov equation. *J. Differential Equations* **129** (1996), 458–508.
- [17] L. A. Peletier and W. C. Troy. *Higher Order Patterns: Higher Order Models in Physics and Mechanics*. Birkhäuser, Basel, 2001.

- [18] D. Sullivan. Cycles for the dynamical study of foliated manifolds and complex manifolds. *Invent. Math.* **36** (1976), 225–255.
- [19] C. Viterbo. A proof of Weinstein’s conjecture in  $\mathbb{R}^{2n}$ . *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4** (1987), 337–356.
- [20] A. Weinstein. On the hypotheses of Rabinowitz’ periodic orbit theorems. *J. Differential Equations* **33** (1979), 353–358.